Harmonic analysis on quantum tori

Quanhua Xu

Wuhan University

and

Université de Besançon

Shanghai - July 2012

Joint work with Zeqian Chen and Yin Zhi

Quantum tori

Let $\theta \in \mathbb{R}$. Let U and V be two unitary operators on a Hilbert space H satisfying the following commutation relation:

$$UV = e^{2\pi i\theta} VU.$$

Example: $H = L_2(\mathbb{T})$ with $\mathbb{T}$ the unit circle; U and V are given:

$$Uf(z) = zf(z) \quad \text{and} \quad Vf(z) = f(e^{-2\pi i\theta} z), \quad f \in L_2(\mathbb{T}),\ z \in \mathbb{T}.$$

Quantum tori

Let $\theta \in \mathbb{R}$. Let U and V be two unitary operators on a Hilbert space H satisfying the following commutation relation:

$$UV = e^{2\pi i\theta} VU.$$

Example: $H = L_2(\mathbb{T})$ with $\mathbb{T}$ the unit circle; U and V are given:

$$Uf(z) = zf(z) \quad \text{and} \quad Vf(z) = f(e^{-2\pi i\theta} z), \quad f \in L_2(\mathbb{T}),\ z \in \mathbb{T}.$$

Let $\mathcal{A}_\theta$ be the universal C*-algebra generated by U and V. This is a quantum (or noncommutative) 2-torus. If θ is irrational, $\mathcal{A}_\theta$ is an irrational rotation C*-algebra. The quantum tori are fundamental examples, probably the most accessible examples for operator algebras and noncommutative geometry.

More generally, let $d \geq 2$ and $\theta = (\theta_{kj})$ be a $d \times d$ real skew-symmetric matrix, i.e. $\theta^{\mathrm{t}} = -\theta$. Let $U_1, \ldots, U_d$ be d unitary operators on H satisfying

$$U_k U_j = e^{2\pi \mathrm{i} \theta_{kj}} U_j U_k, \; j, k = 1, \ldots, d.$$

Let $\mathcal{A}_\theta$ be the universal C*-algebra generated by $U_1, \ldots, U_d$.
This is the noncommutative d-torus associated with θ.
In this talk $U = (U_1, \cdots, U_d)$, θ and $\mathcal{A}_\theta$ will be fixed as above.

More generally, let $d \geq 2$ and $\theta = (\theta_{kj})$ be a $d \times d$ real skew-symmetric matrix, i.e. $\theta^{t} = -\theta$. Let $U_1, \ldots, U_d$ be d unitary operators on H satisfying

$$U_k U_j = e^{2\pi i \theta_{kj}} U_j U_k, \; j, k = 1, \ldots, d.$$

Let $\mathcal{A}_\theta$ be the universal C*-algebra generated by $U_1, \ldots, U_d$.
This is the noncommutative d-torus associated with θ.
In this talk $U = (U_1, \cdots, U_d)$, θ and $\mathcal{A}_\theta$ will be fixed as above.

Notation used throughout the talk:

- Elements of $\mathbb{Z}^d$ are denoted by $m = (m_1, \cdots, m_d)$.
- $\mathbb{T}^d$ is the usual d-torus:

$$\mathbb{T}^d = \left\{(z_1, \ldots, z_d) \; : \; |z_j| = 1, z_j \in \mathbb{C}\right\}$$

- For $m \in \mathbb{Z}^d$ and $z = (z_1, \ldots, z_d) \in \mathbb{T}^d$ let

$$z^m = z_1^{m_1} \cdots z_d^{m_d} \quad \text{and} \quad U^m = U_1^{m_1} \cdots U_d^{m_d},$$

where $U = (U_1, \ldots, U_d)$.

Trace - noncommutative measure

A polynomial in $U = (U_1, \cdots, U_d)$ is a finite sum

$$x = \sum_{m \in \mathbb{Z}^d} \alpha_m U^m \in \mathcal{A}_\theta \quad \text{with} \quad \alpha_m \in \mathbb{C}.$$

Let $\mathcal{P}_\theta$ denote the involutive subalgebra of all such polynomials. Then $\mathcal{P}_\theta$ is dense in $\mathcal{A}_\theta$. For any x as above define

$$\tau(x) = \alpha_{\mathbf{0}} \quad \text{with} \quad \mathbf{0} = (0, \cdots, 0).$$

Then τ extends to a faithful tracial state on $\mathcal{A}_\theta$.

Trace - noncommutative measure

A polynomial in $U = (U_1, \cdots, U_d)$ is a finite sum

$$x = \sum_{m \in \mathbb{Z}^d} \alpha_m U^m \in \mathcal{A}_\theta \quad \text{with} \quad \alpha_m \in \mathbb{C}.$$

Let $\mathcal{P}_\theta$ denote the involutive subalgebra of all such polynomials. Then $\mathcal{P}_\theta$ is dense in $\mathcal{A}_\theta$. For any x as above define

$$\tau(x) = \alpha_{\mathbf{0}} \quad \text{with} \quad \mathbf{0} = (0, \cdots, 0).$$

Then τ extends to a faithful tracial state on $\mathcal{A}_\theta$.
Let $\mathbb{T}_\theta^d$ be the w*-closure of $\mathcal{A}_\theta$ in the GNS representation of τ.
Then τ becomes a normal faithful tracial state on $\mathbb{T}_\theta^d$. Thus $(\mathbb{T}_\theta^d, \tau)$ is a tracial noncommutative probability space.

Noncommutative L_p-spaces

For $1 \leq p < \infty$ and $x \in \mathbb{T}_\theta^d$ let

$$\|x\|_p = \left(\tau(|x|^p)\right)^{\frac{1}{p}} \quad \text{with} \quad |x| = (x^* x)^{\frac{1}{2}}.$$

This defines a norm on $\mathbb{T}_\theta^d$. The corresponding completion is denoted by $L_p(\mathbb{T}_\theta^d)$. We also set $L_\infty(\mathbb{T}_\theta^d) = \mathbb{T}_\theta^d$.

Noncommutative L_p-spaces

For $1 \leq p < \infty$ and $x \in \mathbb{T}_\theta^d$ let

$$\|x\|_p = \left(\tau(|x|^p)\right)^{\frac{1}{p}} \quad \text{with} \quad |x| = (x^* x)^{\frac{1}{2}}.$$

This defines a norm on $\mathbb{T}_\theta^d$. The corresponding completion is denoted by $L_p(\mathbb{T}_\theta^d)$. We also set $L_\infty(\mathbb{T}_\theta^d) = \mathbb{T}_\theta^d$.

The philosophy behind is explained as follows:

$$\text{probability space } (\mathbb{T}^d, \mu) \quad \leftrightarrow \quad \text{noncom probability space} (\mathbb{T}_\theta^d, \tau)$$

Noncommutative L_p-spaces

For $1 \leq p < \infty$ and $x \in \mathbb{T}_\theta^d$ let

$$\|x\|_p = \left(\tau(|x|^p)\right)^{\frac{1}{p}} \quad \text{with} \quad |x| = (x^* x)^{\frac{1}{2}}.$$

This defines a norm on $\mathbb{T}_\theta^d$. The corresponding completion is denoted by $L_p(\mathbb{T}_\theta^d)$. We also set $L_\infty(\mathbb{T}_\theta^d) = \mathbb{T}_\theta^d$.

The philosophy behind is explained as follows:

$$\begin{array}{rcl} \text{probability space } (\mathbb{T}^d, \mu) & \leftrightarrow & \text{noncom probability space} (\mathbb{T}_\theta^d, \tau) \\ \text{commutative algebra } L_\infty(\mathbb{T}^d) & \leftrightarrow & \text{noncommutative algebra } \mathbb{T}_\theta^d \end{array}$$

Noncommutative L_p-spaces

For $1 \le p < \infty$ and $x \in \mathbb{T}_\theta^d$ let

$$\|x\|_p = \big(\tau(|x|^p)\big)^{\frac{1}{p}} \quad \text{with} \quad |x| = (x^*x)^{\frac{1}{2}}.$$

This defines a norm on $\mathbb{T}_\theta^d$. The corresponding completion is denoted by $L_p(\mathbb{T}_\theta^d)$. We also set $L_\infty(\mathbb{T}_\theta^d) = \mathbb{T}_\theta^d$.

The philosophy behind is explained as follows:

$$\begin{aligned} \text{probability space } (\mathbb{T}^d, \mu) \quad &\leftrightarrow \quad \text{noncom probability space} (\mathbb{T}_\theta^d, \tau) \\ \text{commutative algebra } L_\infty(\mathbb{T}^d) \quad &\leftrightarrow \quad \text{noncommutative algebra } \mathbb{T}_\theta^d \\ \text{integration against } \mu \int_{\mathbb{T}^d} \quad &\leftrightarrow \quad \text{trace } \tau \end{aligned}$$

Noncommutative L_p-spaces

For $1 \leq p < \infty$ and $x \in \mathbb{T}_\theta^d$ let

$$\|x\|_p = \left(\tau(|x|^p)\right)^{\frac{1}{p}} \quad \text{with} \quad |x| = (x^* x)^{\frac{1}{2}}.$$

This defines a norm on $\mathbb{T}_\theta^d$. The corresponding completion is denoted by $L_p(\mathbb{T}_\theta^d)$. We also set $L_\infty(\mathbb{T}_\theta^d) = \mathbb{T}_\theta^d$.

The philosophy behind is explained as follows:

$$\begin{array}{rcl}
\text{probability space } (\mathbb{T}^d, \mu) & \leftrightarrow & \text{noncom probability space} (\mathbb{T}_\theta^d, \tau) \\
\text{commutative algebra } L_\infty(\mathbb{T}^d) & \leftrightarrow & \text{noncommutative algebra } \mathbb{T}_\theta^d \\
\text{integration against } \mu \displaystyle\int_{\mathbb{T}^d} & \leftrightarrow & \text{trace } \tau \\
\displaystyle\int_{\mathbb{T}^d} : L_\infty(\mathbb{T}^d) \to \mathbb{C} & \leftrightarrow & \tau : \mathbb{T}_\theta^d \to \mathbb{C}
\end{array}$$

Noncommutative L_p-spaces

For $1 \leq p < \infty$ and $x \in \mathbb{T}_\theta^d$ let

$$\|x\|_p = \big(\tau(|x|^p)\big)^{\frac{1}{p}} \quad \text{with} \quad |x| = (x^* x)^{\frac{1}{2}}.$$

This defines a norm on $\mathbb{T}_\theta^d$. The corresponding completion is denoted by $L_p(\mathbb{T}_\theta^d)$. We also set $L_\infty(\mathbb{T}_\theta^d) = \mathbb{T}_\theta^d$.

The philosophy behind is explained as follows:

$$\begin{aligned}
\text{probability space } (\mathbb{T}^d, \mu) &\quad\leftrightarrow\quad \text{noncom probability space} (\mathbb{T}_\theta^d, \tau) \\
\text{commutative algebra } L_\infty(\mathbb{T}^d) &\quad\leftrightarrow\quad \text{noncommutative algebra } \mathbb{T}_\theta^d \\
\text{integration against } \mu \int_{\mathbb{T}^d} &\quad\leftrightarrow\quad \text{trace } \tau \\
\int_{\mathbb{T}^d} : L_\infty(\mathbb{T}^d) \to \mathbb{C} &\quad\leftrightarrow\quad \tau : \mathbb{T}_\theta^d \to \mathbb{C} \\
\int_{\mathbb{T}^d} f d\mu &\quad\leftrightarrow\quad \tau(x)
\end{aligned}$$

Noncommutative L_p-spaces

For $1 \le p < \infty$ and $x \in \mathbb{T}_\theta^d$ let

$$\|x\|_p = \big(\tau(|x|^p)\big)^{\frac{1}{p}} \quad \text{with} \quad |x| = (x^*x)^{\frac{1}{2}}.$$

This defines a norm on $\mathbb{T}_\theta^d$. The corresponding completion is denoted by $L_p(\mathbb{T}_\theta^d)$. We also set $L_\infty(\mathbb{T}_\theta^d) = \mathbb{T}_\theta^d$.

The philosophy behind is explained as follows:

$$\begin{aligned}
\text{probability space } (\mathbb{T}^d, \mu) \quad &\leftrightarrow \quad \text{noncom probability space}(\mathbb{T}_\theta^d, \tau) \\
\text{commutative algebra } L_\infty(\mathbb{T}^d) \quad &\leftrightarrow \quad \text{noncommutative algebra } \mathbb{T}_\theta^d \\
\text{integration against } \mu \int_{\mathbb{T}^d} \quad &\leftrightarrow \quad \text{trace } \tau \\
\int_{\mathbb{T}^d} : L_\infty(\mathbb{T}^d) \to \mathbb{C} \quad &\leftrightarrow \quad \tau : \mathbb{T}_\theta^d \to \mathbb{C} \\
\int_{\mathbb{T}^d} f d\mu \quad &\leftrightarrow \quad \tau(x) \\
\|f\|_p = \Big(\int_{\mathbb{T}^d} |f|^p d\mu\Big)^{\frac{1}{p}} \quad &\leftrightarrow \quad \|x\|_p = \big(\tau(|x|^p)\big)^{\frac{1}{p}}
\end{aligned}$$

Noncommutative L_p-spaces

For $1 \le p < \infty$ and $x \in \mathbb{T}_\theta^d$ let

$$\|x\|_p = \big(\tau(|x|^p)\big)^{\frac{1}{p}} \quad \text{with} \quad |x| = (x^*x)^{\frac{1}{2}}.$$

This defines a norm on $\mathbb{T}_\theta^d$. The corresponding completion is denoted by $L_p(\mathbb{T}_\theta^d)$. We also set $L_\infty(\mathbb{T}_\theta^d) = \mathbb{T}_\theta^d$.

The philosophy behind is explained as follows:

$$\begin{aligned}
\text{probability space } (\mathbb{T}^d, \mu) \quad &\leftrightarrow \quad \text{noncom probability space} (\mathbb{T}_\theta^d, \tau) \\
\text{commutative algebra } L_\infty(\mathbb{T}^d) \quad &\leftrightarrow \quad \text{noncommutative algebra } \mathbb{T}_\theta^d \\
\text{integration against } \mu \int_{\mathbb{T}^d} \quad &\leftrightarrow \quad \text{trace } \tau \\
\int_{\mathbb{T}^d} : L_\infty(\mathbb{T}^d) \to \mathbb{C} \quad &\leftrightarrow \quad \tau : \mathbb{T}_\theta^d \to \mathbb{C} \\
\int_{\mathbb{T}^d} f d\mu \quad &\leftrightarrow \quad \tau(x) \\
\|f\|_p = \Big(\int_{\mathbb{T}^d} |f|^p d\mu\Big)^{\frac{1}{p}} \quad &\leftrightarrow \quad \|x\|_p = \big(\tau(|x|^p)\big)^{\frac{1}{p}} \\
L_p(\mathbb{T}^d) \quad &\leftrightarrow \quad L_p(\mathbb{T}_\theta^d)
\end{aligned}$$

Fourier coefficients

The trace τ extends to a contractive functional on $L_1(\mathbb{T}_\theta^d)$. Thus given $x \in L_p(\mathbb{T}_\theta^d)$ define

$$\hat{x}(m) = \tau((U^m)^* x) = \alpha_m, \quad m \in \mathbb{Z}^d.$$

These are the Fourier coefficients of x. Like in the classical case we formally write

$$x \sim \sum_{m \in \mathbb{Z}^d} \hat{x}(m) U^m.$$

This is the Fourier series of x; x is uniquely determined by its Fourier series.
We will study various properties of Fourier series like multipliers, mean and pointwise convergence.

Fourier multipliers on the usual d-torus

Let $\phi = \{\phi_m\}_{m\in\mathbb{Z}^d} \subset \mathbb{C}$. Recall that ϕ is a Fourier multiplier on $L_p(\mathbb{T}^d)$ if the map

$$\sum_{m\in\mathbb{Z}^d} \alpha_m z^m \mapsto \sum_{m\in\mathbb{Z}^d} \phi_m \alpha_m z^m$$

is bounded on $L_p(\mathbb{T}^d)$. Let $M(L_p(\mathbb{T}^d))$ denote the space of all Fourier multipliers on $L_p(\mathbb{T}^d)$, equipped with the natural norm.

Fourier multipliers on the usual d-torus

Let $\phi = \{\phi_m\}_{m\in\mathbb{Z}^d} \subset \mathbb{C}$. Recall that ϕ is a Fourier multiplier on $L_p(\mathbb{T}^d)$ if the map

$$\sum_{m\in\mathbb{Z}^d} \alpha_m z^m \mapsto \sum_{m\in\mathbb{Z}^d} \phi_m \alpha_m z^m$$

is bounded on $L_p(\mathbb{T}^d)$. Let $M(L_p(\mathbb{T}^d))$ denote the space of all Fourier multipliers on $L_p(\mathbb{T}^d)$, equipped with the natural norm.

Simple facts.

- $M(L_2(\mathbb{T}^d)) = \ell_\infty(\mathbb{Z}^d)$
- $M(L_p(\mathbb{T}^d)) = M(L_{p'}(\mathbb{T}^d))$ with p' the conjugate index of p
- $\phi \in M(L_1(\mathbb{T}^d))$ iff ϕ is the Fourier transform of a bounded measure, i.e., $\exists\, \mu$, a bounded measure on $\mathbb{T}^d$ s.t. $\widehat{\mu}(m) = \phi_m$ for all $m \in \mathbb{Z}^d$.

Completely bounded multipliers

We will also need completely bounded multipliers. Recall that a map T is completely bounded (cb for short) on $L_p(\mathbb{T}^d)$ if $T \otimes \mathrm{Id}_{S_p}$ is bounded on $L_p(\mathbb{T}^d; S_p)$, where S_p denotes the Schatten p-class. We then set

$$\|T\|_{\mathrm{cb}} = \|T \otimes \mathrm{Id}_{S_p}\|.$$

ϕ is called a cb Fourier multiplier on $L_p(\mathbb{T}^d)$ if T_ϕ is cb on $L_p(\mathbb{T}^d)$. $M_{\mathrm{cb}}(L_p(\mathbb{T}^d))$ denotes the space of all cb Fourier multipliers on $L_p(\mathbb{T}^d)$.

Completely bounded multipliers

We will also need completely bounded multipliers. Recall that a map T is completely bounded (cb for short) on $L_p(\mathbb{T}^d)$ if $T \otimes \mathrm{Id}_{S_p}$ is bounded on $L_p(\mathbb{T}^d; S_p)$, where S_p denotes the Schatten p-class. We then set

$$\|T\|_{\mathrm{cb}} = \|T \otimes \mathrm{Id}_{S_p}\|.$$

ϕ is called a cb Fourier multiplier on $L_p(\mathbb{T}^d)$ if T_ϕ is cb on $L_p(\mathbb{T}^d)$. $M_{\mathrm{cb}}(L_p(\mathbb{T}^d))$ denotes the space of all cb Fourier multipliers on $L_p(\mathbb{T}^d)$.

It is known that

$$M_{\mathrm{cb}}(L_p(\mathbb{T}^d)) = M(L_p(\mathbb{T}^d))$$

for $p \in \{1, 2, \infty\}$, and only for these three values of p.

Fourier multipliers on the quantum torus

Similarly, we define Fourier multipliers on the noncommutative d-torus $\mathbb{T}_\theta^d$.
Again, let $\phi = \{\phi_m\}_{m\in\mathbb{Z}^d} \subset \mathbb{C}$ and

$$T_\phi : \sum_{m\in\mathbb{Z}^d} \alpha_m U^m \mapsto \sum_{m\in\mathbb{Z}^d} \phi_m \alpha_m U^m$$

for any polynomial $x \in \mathcal{P}_\theta$. We call ϕ a Fourier multiplier on $L_p(\mathbb{T}_\theta^d)$ if T_ϕ extends to a bounded map on $L_p(\mathbb{T}_\theta^d)$. Let $M(L_p(\mathbb{T}_\theta^d))$ denote the space of all L_p Fourier multipliers on $\mathbb{T}_\theta^d$.

Fourier multipliers on the quantum torus

Similarly, we define Fourier multipliers on the noncommutative d-torus $\mathbb{T}_\theta^d$.
Again, let $\phi = \{\phi_m\}_{m\in\mathbb{Z}^d} \subset \mathbb{C}$ and

$$T_\phi : \sum_{m\in\mathbb{Z}^d} \alpha_m U^m \mapsto \sum_{m\in\mathbb{Z}^d} \phi_m \alpha_m U^m$$

for any polynomial $x \in \mathcal{P}_\theta$. We call ϕ a Fourier multiplier on $L_p(\mathbb{T}_\theta^d)$ if T_ϕ extends to a bounded map on $L_p(\mathbb{T}_\theta^d)$. Let $M(L_p(\mathbb{T}_\theta^d))$ denote the space of all L_p Fourier multipliers on $\mathbb{T}_\theta^d$.

Similarly, we define cb Fourier multipliers on $L_p(\mathbb{T}_\theta^d)$ and introduce the corresponding space $M_{\mathrm{cb}}(L_p(\mathbb{T}_\theta^d))$.

Recall that T_ϕ is cb on $L_p(\mathbb{T}_\theta^d)$ if $\mathrm{Id} \otimes T$ is bounded on $L_p(B(\ell_2)\bar{\otimes}\mathbb{T}_\theta^d)$.

Theorem. Let $2 \le p \le \infty$. Then

$$M_{\mathrm{cb}}(L_p(\mathbb{T}_\theta^d)) = M_{\mathrm{cb}}(L_p(\mathbb{T}^d)).$$

Theorem. Let $2 \leq p \leq \infty$. Then

$$M_{\mathrm{cb}}(L_p(\mathbb{T}^d_\theta)) = M_{\mathrm{cb}}(L_p(\mathbb{T}^d)).$$

Proof. Easy direction (independently by Junge, Mei, Parcet):

$$M_{\mathrm{cb}}(L_p(\mathbb{T}^d)) \subset M_{\mathrm{cb}}(L_p(\mathbb{T}^d_\theta)).$$

The tool is transference.

Theorem. Let $2 \le p \le \infty$. Then

$$M_{\mathrm{cb}}(L_p(\mathbb{T}_\theta^d)) = M_{\mathrm{cb}}(L_p(\mathbb{T}^d)).$$

Proof. Easy direction (independently by Junge, Mei, Parcet):

$$M_{\mathrm{cb}}(L_p(\mathbb{T}^d)) \subset M_{\mathrm{cb}}(L_p(\mathbb{T}_\theta^d)).$$

The tool is transference. Let $\phi \in M_{\mathrm{cb}}(L_p(\mathbb{T}^d))$ and $x \in L_p(B(\ell_2)\bar{\otimes}\mathbb{T}_\theta^d)$. For $z \in \mathbb{T}^d$ define

$$\pi_z(x) = \sum_{m \in \mathbb{Z}^d} \hat{x}(m) \otimes U^m z^m.$$

Then π_z is an isometry on $L_p(B(\ell_2)\bar{\otimes}\mathbb{T}_\theta^d)$.

Theorem. Let $2 \leq p \leq \infty$. Then

$$M_{\mathrm{cb}}(L_p(\mathbb{T}_\theta^d)) = M_{\mathrm{cb}}(L_p(\mathbb{T}^d)).$$

Proof. Easy direction (independently by Junge, Mei, Parcet):

$$M_{\mathrm{cb}}(L_p(\mathbb{T}^d)) \subset M_{\mathrm{cb}}(L_p(\mathbb{T}_\theta^d)).$$

The tool is transference. Let $\phi \in M_{\mathrm{cb}}(L_p(\mathbb{T}^d))$ and $x \in L_p(B(\ell_2)\bar{\otimes}\mathbb{T}_\theta^d)$. For $z \in \mathbb{T}^d$ define

$$\pi_z(x) = \sum_{m \in \mathbb{Z}^d} \hat{x}(m) \otimes U^m z^m.$$

Then π_z is an isometry on $L_p(B(\ell_2)\bar{\otimes}\mathbb{T}_\theta^d)$. We consider $z \mapsto \pi_z(x)$ as a function from $\mathbb{T}^d$ to $L_p(B(\ell_2)\bar{\otimes}\mathbb{T}_\theta^d)$. Then

$$\|\pi_\cdot(x)\|_{L_p(\mathbb{T}^d; L_p(B(\ell_2)\bar{\otimes}\mathbb{T}_\theta^d))} = \|x\|_{L_p(B(\ell_2)\bar{\otimes}\mathbb{T}_\theta^d)}.$$

Theorem. Let $2 \leq p \leq \infty$. Then

$$M_{\rm cb}(L_p(\mathbb{T}^d_\theta)) = M_{\rm cb}(L_p(\mathbb{T}^d)).$$

Proof. Easy direction (independently by Junge, Mei, Parcet):

$$M_{\rm cb}(L_p(\mathbb{T}^d)) \subset M_{\rm cb}(L_p(\mathbb{T}^d_\theta)).$$

The tool is transference. Let $\phi \in M_{\rm cb}(L_p(\mathbb{T}^d))$ and $x \in L_p(B(\ell_2)\bar{\otimes}\mathbb{T}^d_\theta)$. For $z \in \mathbb{T}^d$ define

$$\pi_z(x) = \sum_{m\in\mathbb{Z}^d} \hat{x}(m) \otimes U^m z^m.$$

Then π_z is an isometry on $L_p(B(\ell_2)\bar{\otimes}\mathbb{T}^d_\theta)$. We consider $z \mapsto \pi_z(x)$ as a function from $\mathbb{T}^d$ to $L_p(B(\ell_2)\bar{\otimes}\mathbb{T}^d_\theta)$. Then

$$\|\pi_\cdot(x)\|_{L_p(\mathbb{T}^d;L_p(B(\ell_2)\bar{\otimes}\mathbb{T}^d_\theta))} = \|x\|_{L_p(B(\ell_2)\bar{\otimes}\mathbb{T}^d_\theta)}.$$

On the other hand,

$$\pi_z(T_\phi(x)) = T_\phi(\pi_z(x)).$$

Here T_ϕ on the left is the Fourier multiplier on $L_p(\mathbb{T}^d_\theta)$ while T_ϕ on the right is the Fourier multiplier on $L_p(\mathbb{T}^d)$.
It then follows that ϕ is a cb Fourier multiplier on $L_p(\mathbb{T}^d_\theta)$.

Hard direction: $M_{\mathrm{cb}}(L_p(\mathbb{T}^d)) \supset M_{\mathrm{cb}}(L_p(\mathbb{T}_\theta^d))$.

Hard direction: $M_{\mathrm{cb}}(L_p(\mathbb{T}^d)) \supset M_{\mathrm{cb}}(L_p(\mathbb{T}_\theta^d))$.

An infinite complex matrix $\alpha = (\alpha_{m,n})_{m,n\in\mathbb{Z}^d}$ indexed by $\mathbb{Z}^d$ is called a Schur multiplier on $S_p(\ell_2(\mathbb{Z}^d))$ if the map $T_\alpha : (a_{m,n})_{m,n\in\mathbb{Z}^d} \mapsto (\alpha_{m,n}a_{m,n})_{m,n\in\mathbb{Z}^d}$ is bounded on $S_p(\ell_2(\mathbb{Z}^d))$.

Hard direction: $M_{\mathrm{cb}}(L_p(\mathbb{T}^d)) \supset M_{\mathrm{cb}}(L_p(\mathbb{T}_\theta^d))$.
An infinite complex matrix $\alpha = (\alpha_{m,n})_{m,n\in\mathbb{Z}^d}$ indexed by $\mathbb{Z}^d$ is called a Schur multiplier on $S_p(\ell_2(\mathbb{Z}^d))$ if the map $T_\alpha : (a_{m,n})_{m,n\in\mathbb{Z}^d} \mapsto (\alpha_{m,n}a_{m,n})_{m,n\in\mathbb{Z}^d}$ is bounded on $S_p(\ell_2(\mathbb{Z}^d))$.
Let $A = (a_{m,n})_{m,n\in\mathbb{Z}^d} \in S_p(\ell_2(\mathbb{Z}^d))$. Define

$$\pi_U(A) = \mathrm{diag}(U^m)_{m\in\mathbb{Z}^d}\, A\, \mathrm{diag}(U^{-m})_{m\in\mathbb{Z}^d}.$$

Hard direction: $M_{\mathrm{cb}}(L_p(\mathbb{T}^d)) \supset M_{\mathrm{cb}}(L_p(\mathbb{T}^d_\theta))$.

An infinite complex matrix $\alpha = (\alpha_{m,n})_{m,n\in\mathbb{Z}^d}$ indexed by $\mathbb{Z}^d$ is called a Schur multiplier on $S_p(\ell_2(\mathbb{Z}^d))$ if the map $T_\alpha : (a_{m,n})_{m,n\in\mathbb{Z}^d} \mapsto (\alpha_{m,n}a_{m,n})_{m,n\in\mathbb{Z}^d}$ is bounded on $S_p(\ell_2(\mathbb{Z}^d))$. Let $A = (a_{m,n})_{m,n\in\mathbb{Z}^d} \in S_p(\ell_2(\mathbb{Z}^d))$. Define

$$\pi_U(A) = \mathrm{diag}(U^m)_{m\in\mathbb{Z}^d}\, A\, \mathrm{diag}(U^{-m})_{m\in\mathbb{Z}^d}.$$

Then

$$\|\pi_U(A)\|_{L_p(B(\ell_2)\bar{\otimes}\mathbb{T}^d_\theta)} = \|A\|_{S_p(\ell_2(\mathbb{Z}^d))}.$$

Hard direction: $M_{\mathrm{cb}}(L_p(\mathbb{T}^d)) \supset M_{\mathrm{cb}}(L_p(\mathbb{T}^d_\theta))$.
An infinite complex matrix $\alpha = (\alpha_{m,n})_{m,n\in\mathbb{Z}^d}$ indexed by $\mathbb{Z}^d$ is called a Schur multiplier on $S_p(\ell_2(\mathbb{Z}^d))$ if the map $T_\alpha : (a_{m,n})_{m,n\in\mathbb{Z}^d} \mapsto (\alpha_{m,n}a_{m,n})_{m,n\in\mathbb{Z}^d}$ is bounded on $S_p(\ell_2(\mathbb{Z}^d))$.
Let $A = (a_{m,n})_{m,n\in\mathbb{Z}^d} \in S_p(\ell_2(\mathbb{Z}^d))$. Define

$$\pi_U(A) = \mathrm{diag}(U^m)_{m\in\mathbb{Z}^d}\, A\, \mathrm{diag}(U^{-m})_{m\in\mathbb{Z}^d}.$$

Then

$$\|\pi_U(A)\|_{L_p(B(\ell_2)\bar{\otimes}\mathbb{T}^d_\theta)} = \|A\|_{S_p(\ell_2(\mathbb{Z}^d))}.$$

Let $\phi \in M_{\mathrm{cb}}(L_p(\mathbb{T}^d_\theta))$ and $\alpha = (\phi(m-n))_{m,n\in\mathbb{Z}^d}$. It is easy to check that

$$T_\phi(\pi_U(A)) = \pi_U(T_\alpha(A)).$$

Whence α is a Schur multiplier on $S_p(\ell_2(\mathbb{Z}^d))$.

Hard direction: $M_{\rm cb}(L_p(\mathbb{T}^d)) \supset M_{\rm cb}(L_p(\mathbb{T}^d_\theta))$.
An infinite complex matrix $\alpha = (\alpha_{m,n})_{m,n\in\mathbb{Z}^d}$ indexed by $\mathbb{Z}^d$ is called a Schur multiplier on $S_p(\ell_2(\mathbb{Z}^d))$ if the map $T_\alpha : (a_{m,n})_{m,n\in\mathbb{Z}^d} \mapsto (\alpha_{m,n}a_{m,n})_{m,n\in\mathbb{Z}^d}$ is bounded on $S_p(\ell_2(\mathbb{Z}^d))$.
Let $A = (a_{m,n})_{m,n\in\mathbb{Z}^d} \in S_p(\ell_2(\mathbb{Z}^d))$. Define

$$\pi_U(A) = \mathrm{diag}(U^m)_{m\in\mathbb{Z}^d}\, A\, \mathrm{diag}(U^{-m})_{m\in\mathbb{Z}^d}\,.$$

Then

$$\|\pi_U(A)\|_{L_p(B(\ell_2)\bar{\otimes}\mathbb{T}^d_\theta)} = \|A\|_{S_p(\ell_2(\mathbb{Z}^d))}.$$

Let $\phi \in M_{\rm cb}(L_p(\mathbb{T}^d_\theta))$ and $\alpha = (\phi(m-n))_{m,n\in\mathbb{Z}^d}$. It is easy to check that

$$T_\phi(\pi_U(A)) = \pi_U(T_\alpha(A)).$$

Whence α is a Schur multiplier on $S_p(\ell_2(\mathbb{Z}^d))$. Considering matrices A with entries in S_p, we prove in the same way that α is cb.

Hard direction: $M_{\mathrm{cb}}(L_p(\mathbb{T}^d)) \supset M_{\mathrm{cb}}(L_p(\mathbb{T}_\theta^d))$.
An infinite complex matrix $\alpha = (\alpha_{m,n})_{m,n\in\mathbb{Z}^d}$ indexed by $\mathbb{Z}^d$ is called a Schur multiplier on $S_p(\ell_2(\mathbb{Z}^d))$ if the map $T_\alpha : (a_{m,n})_{m,n\in\mathbb{Z}^d} \mapsto (\alpha_{m,n}a_{m,n})_{m,n\in\mathbb{Z}^d}$ is bounded on $S_p(\ell_2(\mathbb{Z}^d))$.
Let $A = (a_{m,n})_{m,n\in\mathbb{Z}^d} \in S_p(\ell_2(\mathbb{Z}^d))$. Define

$$\pi_U(A) = \mathrm{diag}(U^m)_{m\in\mathbb{Z}^d}\, A\, \mathrm{diag}(U^{-m})_{m\in\mathbb{Z}^d}.$$

Then

$$\|\pi_U(A)\|_{L_p(B(\ell_2)\bar{\otimes}\mathbb{T}_\theta^d)} = \|A\|_{S_p(\ell_2(\mathbb{Z}^d))}.$$

Let $\phi \in M_{\mathrm{cb}}(L_p(\mathbb{T}_\theta^d))$ and $\alpha = (\phi(m-n))_{m,n\in\mathbb{Z}^d}$. It is easy to check that

$$T_\phi(\pi_U(A)) = \pi_U(T_\alpha(A)).$$

Whence α is a Schur multiplier on $S_p(\ell_2(\mathbb{Z}^d))$. Considering matrices A with entries in S_p, we prove in the same way that α is cb. Then it is well known that ϕ is a cb multiplier on $L_p(\mathbb{T}^d)$ for $p = \infty$. By a very recent transference theorem of Neuwirth-Ricard, this latter result remains true for $p < \infty$. Thus $\phi \in M_{\mathrm{cb}}(L_p(\mathbb{T}^d))$.

Summation methods

Let $x \in L_p(\mathbb{T}_\theta^d)$ with $1 \leq p \leq \infty$.

- Square Fejer means:

$$F_n[x] = \sum_{m \in \mathbb{Z}^d, |m_j| \leq n} \left(1 - \frac{|m_1|}{n+1}\right) \cdots \left(1 - \frac{|m_d|}{n+1}\right) \hat{x}(m) U^m$$

- Circular Poisson means:

$$\mathbb{P}_r(x) = \sum_{m \in \mathbb{Z}^d} \hat{x}(m) r^{|m|_2} U^m$$

where $|m|_2 = (|m_1|^2 + \cdots + |m_d|^2)^{1/2}$.

Fundamental problem: In which sense do these means of x converge back to x?

Mean convergence

Proposition (mean convergence theorem).
Let $1 \leq p < \infty$. If $x \in L_p(\mathbb{T}_\theta^d)$ then

$$\lim_{n\to\infty} F_n[x] = \lim_{r\to\infty} \mathbb{P}_r[x] = x \text{ in } L_p(\mathbb{T}_\theta^d).$$

Mean convergence

Proposition (mean convergence theorem).
Let $1 \leq p < \infty$. If $x \in L_p(\mathbb{T}_\theta^d)$ then

$$\lim_{n\to\infty} F_n[x] = \lim_{r\to\infty} \mathbb{P}_r[x] = x \ \text{ in } \ L_p(\mathbb{T}_\theta^d).$$

But the problem for the pointwise convergence is hard and delicate for several raisons:

- We are dealing with operators instead of functions.
- Usually in the commutative case, a pointwise convergence theorem is based on the corresponding mean theorem and maximal inequality.

Pointwise convergence

Definition (C. Lance)
A sequence (x_n) in $L_p(\mathbb{T}_\theta^d)$ is said to converge bilaterally almost uniformly (b.a.u.) to x if for any $\varepsilon > 0$ there is a projection $e \in \mathbb{T}_\theta^d$ s.t.

$$\tau(1-e) < \varepsilon \quad \text{and} \quad \lim_{n\to\infty} \|e(x_n - x)e\|_\infty = 0.$$

Remark. In the commutative case this is equivalent to the almost everywhere convergence (Egorov's theorem).

Question. Let $1 \le p \le \infty$ and $x \in L_p(\mathbb{T}_\theta^d)$. Do we have

$$F_n[x] \xrightarrow{b.a.u} x \text{ as } n \to \infty \quad \text{and} \quad \mathbb{P}_r[x] \xrightarrow{b.a.u} x \text{ as } r \to \infty\,?$$

Maximal inequalities

This is a subtle part of the talk. We don't have the noncommutative analogue of the usual pointwise maximal function. Even for any positive 2×2-matrices a, b,

$max(a, b)$ **does not make any sense**

Maximal inequalities

This is a subtle part of the talk. We don't have the noncommutative analogue of the usual pointwise maximal function. Even for any positive 2×2-matrices a, b,

$$max(a,b) \textbf{ does not make any sense}$$

Instead, we define the space $L_p(\mathbb{T}_\theta^d;\ell_\infty)$. For a sequence $x = (x_n)$ of positive operators in $L_p(\mathbb{T}_\theta^d)$ we define x to be in $L_p(\mathbb{T}_\theta^d;\ell_\infty)$ if there is a positive $a \in L_p(\mathbb{T}_\theta^d)$ s.t.

$$x_n \le a, \quad \forall\, n \in \mathbb{N}.$$

Then $\|x\|_{L_p(\mathbb{T}_\theta^d;\ell_\infty)}$ is defined to be $\inf \|a\|_p$.

Remark. We skip the definition of $\|x\|_{L_p(\mathbb{T}_\theta^d;\ell_\infty)}$ for a general x. This norm is denoted by $\|\sup_n^+ x_n\|_p$. Note that this is only a notation since $\sup x_n$ does not make any sense in the noncommutative setting.

Theorem (maximal inequalities): $1 < p \leq \infty$, $x \in L_p(\mathbb{T}_\theta^d)$. Then

$$\|\sup_{n\geq 1}{}^{+} F_n[x]\|_p \leq C_p \|x\|_p \quad \text{and} \quad \|\sup_{r>0}{}^{+} \mathbb{P}_r[x]\|_p \leq C_p \|x\|_p.$$

In particular, if x is positive, then there is $a \in L_p(\mathbb{T}_\theta^d)$ s.t.

$$\|a\|_p \leq C_p \|x\|_p$$

and

$$F_n[x] \leq a, \quad \forall\, n \geq 1 \quad \text{and} \quad \mathbb{P}_r[x] \leq a, \quad \forall\, 0 \leq r < 1.$$

For $p = 1$ we have a weak type (1, 1) substitute.

Theorem (maximal inequalities): $1 < p \le \infty$, $x \in L_p(\mathbb{T}_\theta^d)$. Then

$$\|\sup_{n\ge1}{}^+ F_n[x]\|_p \le C_p\|x\|_p \quad \text{and} \quad \|\sup_{r>0}{}^+ \mathbb{P}_r[x]\|_p \le C_p\|x\|_p.$$

In particular, if x is positive, then there is $a \in L_p(\mathbb{T}_\theta^d)$ s.t.

$$\|a\|_p \le C_p\|x\|_p$$

and

$$F_n[x] \le a, \quad \forall\, n \ge 1 \quad \text{and} \quad \mathbb{P}_r[x] \le a, \quad \forall\, 0 \le r < 1.$$

For $p = 1$ we have a weak type (1, 1) substitute.
Idea of proof. $(\mathbb{P}_r)_{0\le r<1}$ is a semigroup of trace preserving positive maps. Applying the noncommutative maximal ergodic inequality (Junge-Xu), we get the maximal inequality for $\mathbb{P}_r$. The proof for the Fejer means $F_n[x]$ uses transference and Tao Mei's noncommutative Hardy-Littlewood maximal inequality. The case $p = 1$ is much harder.

Theorem (maximal inequalities): $1 < p \leq \infty$, $x \in L_p(\mathbb{T}_\theta^d)$. Then

$$\|\sup_{n\geq 1}{}^{+} F_n[x]\|_p \leq C_p \|x\|_p \quad \text{and} \quad \|\sup_{r>0}{}^{+} \mathbb{P}_r[x]\|_p \leq C_p \|x\|_p .$$

In particular, if x is positive, then there is $a \in L_p(\mathbb{T}_\theta^d)$ s.t.

$$\|a\|_p \leq C_p \|x\|_p$$

and

$$F_n[x] \leq a, \quad \forall\, n \geq 1 \quad \text{and} \quad \mathbb{P}_r[x] \leq a, \quad \forall\, 0 \leq r < 1.$$

For $p = 1$ we have a weak type (1, 1) substitute.
Idea of proof. $(\mathbb{P}_r)_{0\leq r<1}$ is a semigroup of trace preserving positive maps. Applying the noncommutative maximal ergodic inequality (Junge-Xu), we get the maximal inequality for $\mathbb{P}_r$. The proof for the Fejer means $F_n[x]$ uses transference and Tao Mei's noncommutative Hardy-Littlewood maximal inequality.
The case $p = 1$ is much harder.
Corollary. Let $1 \leq p \leq \infty$ and $x \in L_p(\mathbb{T}_\theta^d)$. Then

$$F_n[x] \xrightarrow{b.a.u} x \ \text{ as } \ n \to \infty \quad \text{and} \quad \mathbb{P}_r[x] \xrightarrow{b.a.u} x \ \text{ as } \ r \to \infty .$$

Square function inequalities

For $x \in L_p(\mathbb{T}_\theta^d)$ we define Littlewood-Paley g-functions

$$G_c(x) = \Big(\int_0^1 \Big|\frac{d}{dr}\mathbb{P}_r[x]\Big|^2 (1-r)dr\Big)^{1/2} \text{ and } G_r(x) = G_c(x^*),$$

where $\mathbb{P}_r$ denotes the circular Poisson means:

$$\mathbb{P}_r[x] = \sum_{m \in \mathbb{Z}^d} \hat{x}(m) r^{|m|_2} U^m$$

Square function inequalities

For $x \in L_p(\mathbb{T}^d_\theta)$ we define Littlewood-Paley g-functions

$$G_c(x) = \Big(\int_0^1 \Big|\frac{d}{dr}\mathbb{P}_r[x]\Big|^2 (1-r)dr\Big)^{1/2} \text{ and } G_r(x) = G_c(x^*),$$

where $\mathbb{P}_r$ denotes the circular Poisson means:

$$\mathbb{P}_r[x] = \sum_{m\in\mathbb{Z}^d} \hat{x}(m) r^{|m|_2} U^m$$

Theorem. Let $2 \le p < \infty$. Then

$$\|x\|_p \approx \max\big(\|G_c(x)\|_p, \ \|G_r(x)\|_p\big).$$

Idea of proof. Use square function inequalities for general quantum semigroups of Junge-Le Merdy-Xu.

Square function inequalities

For $x \in L_p(\mathbb{T}_\theta^d)$ we define Littlewood-Paley g-functions

$$G_c(x) = \Big(\int_0^1 \Big|\frac{d}{dr}\mathbb{P}_r[x]\Big|^2 (1-r)dr\Big)^{1/2} \text{ and } G_r(x) = G_c(x^*),$$

where $\mathbb{P}_r$ denotes the circular Poisson means:

$$\mathbb{P}_r[x] = \sum_{m\in\mathbb{Z}^d} \hat{x}(m) r^{|m|_2} U^m$$

Theorem. Let $2 \le p < \infty$. Then

$$\|x\|_p \approx \max\Big(\|G_c(x)\|_p,\ \|G_r(x)\|_p\Big).$$

Idea of proof. Use square function inequalities for general quantum semigroups of Junge-Le Merdy-Xu.

Remark. 1) A similar inequality for $1 < p < 2$ by replacing max by an inf.

2) For $p = 1$ we can introduce the corresponding Hardy space H_1 and describe its dual space as a BMO space.